

A Nonlinear Fuzzy PID Controller via Algebraic Product AND - Bounded Sum OR - Algebraic Product Inference

B. M. Mohan, NeethuKuruville

Abstract—This paper reveals a mathematical model of the simplest fuzzy PID controller which employs two fuzzy sets (negative and positive) on each of the three input variables (displacement, velocity and acceleration) and four fuzzy sets (-2, -1, +1, +2) on the output variable (incremental control). L-type, Γ -type and Π -type membership functions are considered in fuzzification process of input and output variables. Controller modeling is done via algebraic product AND operator-bounded sum OR operator-algebraic product inference method-Center of Sums defuzzification process combination. The model obtained in this manner, turns out to be nonlinear, is analysed finally.

Index Terms—PID control, nonlinear control, fuzzy control, mathematical modeling, algebraic product, bounded sum, center of sums (CoS).

1 INTRODUCTION

CONVENTIONAL (linear) PID controllers have been in extensive usage in industrial automation and process control. The reason behind this is their ease of design, low cost, simplicity of operation, inexpensive maintenance and effectiveness for most linear systems. These controllers generally do not work well for nonlinear systems, higher order linear systems and systems which are complex and vague having no precise mathematical models. To overcome this difficulty, various types of modified linear PID controllers such as auto tuned and adaptive PID controllers have been developed. Alternatively, controllers employing fuzzy logic have also been implemented sometimes.

Let us now take a look at the historical developments in fuzzy control technology. A fuzzy PID controller has been constructed (Wang and Kwok, 1992) by combining a fuzzy PD controller and a fuzzy I controller in parallel. It has been shown (Mizumoto, 1995) that PID controllers can be realized by product-sum-gravity method and simplified fuzzy reasoning method. A new fuzzy PID controller structure has been proposed (Qiao and Mizumoto, 1996) and a parameter adaptive method via peak observer has been presented to tune the

parameters of the fuzzy controller on-line. Analytical structure for a fuzzy PID controller and its BIBO stability analysis has been studied (Misir et al, 1996). Fuzzy PI and fuzzy PD controllers have been combined in cascade to get a fuzzy PID controller (Kim and Oh, 2000).

In order to improve the performance in transient and steady

states, an adaptive method via function tuner has been developed (Woo et al, 2000) to tune the scaling factors of the fuzzy controller online. A tuning method, based on gain margin and phase margin specifications, has been proposed (Xu et al, 2000) for determining the parameters of the fuzzy PID controller. Several forms of decomposed PID fuzzy logic controllers have been tested and compared (Golob, 2001). A function based evaluation has been proposed (Hu et al, 2001) for a systematic study of fuzzy PID controllers. An adaptive method via relative rate observer has been proposed (Guzelkaya et al, 2003) for tuning the scaling factors of the fuzzy logic controller in an on-line manner.

Mohan and Sinha (2006) have shown that algebraic product triangular norm - bounded sum triangular co-norm - algebraic product inference method - CoSdefuzzification method combination leads to a linear fuzzy PID controller. Mohan and Sinha (2008a) have shown that the analytical structures of fuzzy PID controllers derived via minimum triangular norm are not suitable for control. Mohan and Sinha (2008b) have introduced an analytical structure for fuzzy PID controller by employing algebraic product triangular norm, bounded sum triangular co-norm, minimum inference method and CoSdefuzzification method. They also have derived conditions for BIBO stability using Small Gain theorem. Since linear controllers are no better than nonlinear controllers in controlling nonlinear and complex plants, in this paper an attempt is made to derive a nonlinear fuzzy PID controller using the same combination (i.e., algebraic product AND-bounded sum OR-algebraic product inference-CoSdefuzzification) and modified output membership functions.

The paper is organized as follows: The next section deals with fundamental components of a typical fuzzy PID controller. Section 3 presents a mathematical model of nonlinear fuzzy PID controller. Properties of the model are

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discussed in Section 4. Section 5 concludes the paper.

2 COMPONENTS OF FUZZY PID CONTROLLER

The incremental control signal generated by a discrete time PID controller is given by:

$$\Delta u(k) = u(k) - u(k-1) = K_P^d v(k) + K_I^d d(k) + K_D^d a(k) \quad (1)$$

where K_P^d , K_I^d and K_D^d are respectively the proportional, integral and derivative constants of a digital PID controller and velocity $v(k)$, displacement $d(k)$ and acceleration $a(k)$ are

given by:

$$v(k) = \{d(k) - d[(k-1)]\} / T(2)$$

$$d(k) = e(k) \quad (3)$$

$$a(k) = \{v(k) - v[(k-1)]\} / T \quad (4)$$

$e(k)$ is the error signal, $r(k)$ is the reference command, $y(k)$ is the plant output, k is the sampling instant, and T is the sampling period. Eq.(1) is known as 'velocity algorithm' and is widely used. The principal structure of a fuzzy PID control, see Fig.1, consists of the following components:

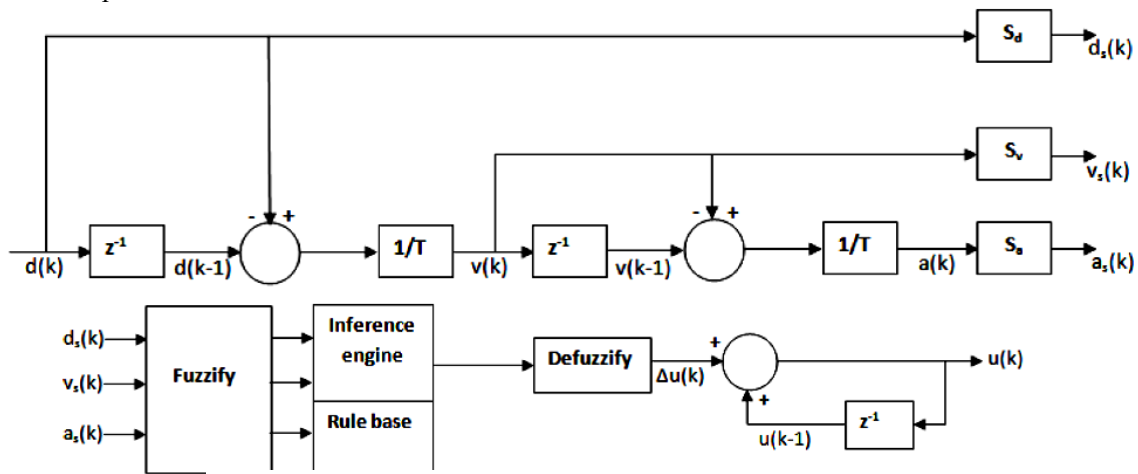


Fig. 1. A typical fuzzy PID controller

2.1 Scaling Factors

Let $d_s(k)$, $v_s(k)$ and $a_s(k)$ be the scaled versions of $d(k)$, $v(k)$ and $a(k)$ respectively. Then

$$d_s(k) = S_d d(k) \quad (5)$$

$$v_s(k) = S_v v(k) \quad (6)$$

$$a_s(k) = S_a a(k) \quad (7)$$

where S_d , S_v and S_a are the scaling factors which play a role similar to that of K_P^d , K_I^d and K_D^d in a conventional PID controller.

2.2 Fuzzification

Fuzzification process converts crisp values of controller scaled inputs into fuzzy sets that can be used by the inference engine (refer Section 2.4) to activate and apply the control rule.

As shown in Fig.1, fuzzy PID controller has three scaled inputs: $d_s(k)$, $v_s(k)$ and $a_s(k)$. These inputs are fuzzified by L-type and Γ -type membership functions as illustrated in Fig. 2. These membership functions on $d_s(k)$ are defined by:

$$\mu_{-D} = \begin{cases} 0 & -L_d \leq d_s \leq -l_d \\ \frac{(-d_s + l_d)}{2l_d} & -l_d \leq d_s \leq l_d \\ 1 & l_d \leq d_s \leq L_d \end{cases} \quad (8)$$

$$\mu_{+D} = \begin{cases} 0 & -L_d \leq d_s \leq -l_d \\ \frac{(d_s + l_d)}{2l_d} & -l_d \leq d_s \leq l_d \\ 1 & l_d \leq d_s \leq L_d \end{cases} \quad (9)$$

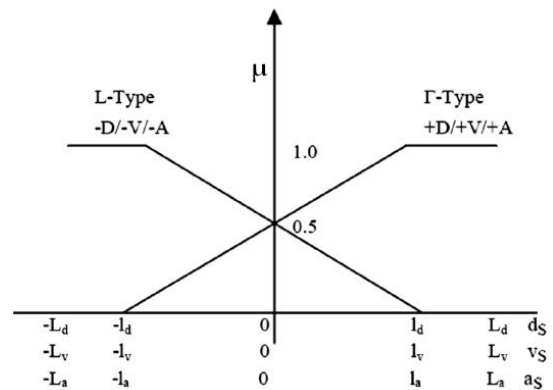


Fig. 2. Input membership functions

It may be noted that (10)

$$\mu_{+D} + \mu_{-D} = 1$$

Membership functions on $v_s(k)$ and $a_s(k)$ can be defined in a similar manner. The membership functions (*L-type*, *Π -type* and *Γ -type*) on $\Delta u(k)$ are shown in Fig. 3. The constants l_d , l_v , l_a , L_d , L_v , L_a , l_u , L_u and w are to be chosen by the designer.

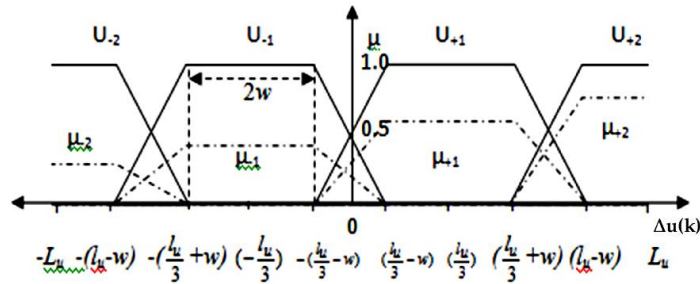


Fig. 3. Output trapezoidal membership functions (reference fuzzy sets: U_{-2} , U_{-1} , U_{+1} , U_{+2} ; inferred fuzzy sets: u_{-2} , u_{-1} , u_{+1} , u_{+2} obtained via algebraic product inference)

2.3 Control Rule Base

The following control rules are considered in terms of the aforementioned input and output fuzzy sets:

R1: If d_s is $-D$ & v_s is $-V$ & a_s is $-A$ then Δu is U_{-2}

R2: If (d_s is $-D$ & v_s is $-V$ & a_s is $+A$) /
 (d_s is $-D$ & v_s is $+V$ & a_s is $-A$) /
 (d_s is $+D$ & v_s is $-V$ & a_s is $-A$) then Δu is U_{-1}

R3: If (d_s is $-D$ & v_s is $+V$ & a_s is $+A$) /
 (d_s is $+D$ & v_s is $-V$ & a_s is $+A$) /
 (d_s is $+D$ & v_s is $+V$ & a_s is $-A$) then Δu is U_{+1}

R4: If d_s is $+D$ & v_s is $+V$ & a_s is $+A$ then Δu is U_{+2}

where the & symbol represents algebraic product AND operation which is defined as:

$$\mu_i(d_s) \& \mu_j(v_s) \& \mu_k(a_s) = \mu_i(d_s) \cdot \mu_j(v_s) \cdot \mu_k(a_s) \quad (11)$$

With $i \in \{-D, +D\}$, $j \in \{-V, +V\}$ and $k \in \{-A, +A\}$. The / symbol in rules R2 and R3 represents bounded sum OR operation defined as

$$\begin{aligned} & \mu_1(d_s, v_s, a_s) / \mu_2(d_s, v_s, a_s) / \mu_3(d_s, v_s, a_s) = \\ & \min \{1, \mu_1(d_s, v_s, a_s) + \mu_2(d_s, v_s, a_s) + \mu_3(d_s, v_s, a_s)\} = \\ & \mu_1(d_s, v_s, a_s) + \mu_2(d_s, v_s, a_s) + \mu_3(d_s, v_s, a_s) \quad (12) \end{aligned}$$

Fig.4. Regions of input

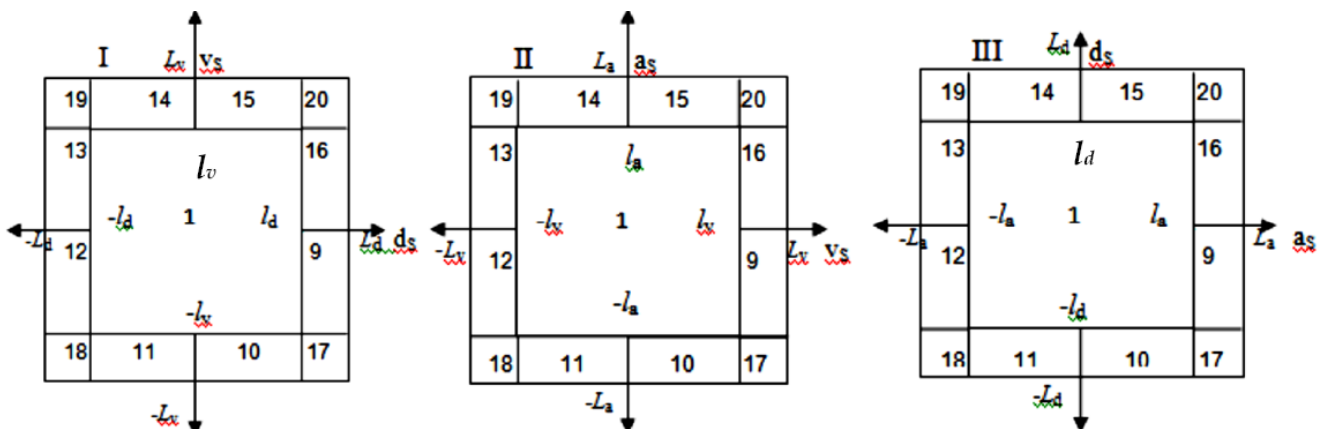
Notice that the control rules are nonlinear as the output fuzzy sets are not linearly related to the input fuzzy sets.

2.4 Inference Engine

Overall value of incremental control output variable is computed by the inference engine by considering the individual contribution of each rule in the rule base. For this, corresponding to each rule, first the degree of match from the crisp input values is found by using the algebraic product AND operator. Then the degree of match is used to determine the inferred output fuzzy set using algebraic product inference method, see Fig. 3.

We consider all possible combinations of these variables in a 3D space. A point, say (x_i, y_i, z_i) , in a 3D space can always be distinctly shown by taking its projection on the xy -, yz - and zx - planes. So, as shown in Fig. 4, thirteen input combinations are considered in each input (d_s - v_s -, v_s - a_s - and a_s - d_s -) plane so that the input point (d_s^*, v_s^*, a_s^*) can be uniquely located in the 3D cell (subspace) represented by the triplet (n_I, n_{II}, n_{III}) where $n_I, n_{II}, n_{III} = 1, 9, 10, \dots, 20$. For example, the triplet $(9, 11, 19)$ represents the 3D cell with 9 from I, 11 from II, and 19 from III of Fig. 4.

The control rules R1 to R4 are used to evaluate appropriate control law in each valid cell (n_I, n_{II}, n_{III}) . By using the algebraic product triangular norm and the bounded sum triangular co-norm, the outcome of premise part of each rule is found in each valid cell and is shown in Table.1. A cell is said to be valid if and only if the relations between d_s and v_s , and a_s and v_s produce the relation between a_s and d_s . For example, the cell $(9, 11, 19)$ is a valid cell because the relations $l_d \leq d_s \leq L_d, -l_v \leq v_s \leq 0, -L_a \leq a_s \leq -l_a$ produce the relations $-L_a \leq a_s \leq -l_a$ and $l_d \leq d_s \leq L_d$.



2.5 Defuzzification

Defuzzification process converts fuzzy information into crisp information. According to CoS method, the incremental control output is given by

$$\Delta u(k) = \frac{A(\mu_{-2})C(\mu_{-2}) + A(\mu_{-1})C(\mu_{-1}) + A(\mu_{+1})C(\mu_{+1}) + A(\mu_{+2})C(\mu_{+2})}{A(\mu_{-2}) + A(\mu_{-1}) + A(\mu_{+1}) + A(\mu_{+2})} \quad (13)$$

where $A()$ and $C()$ are respectively the area and centre of area of the inferred output fuzzy set. From Fig. 3 we have

$$\begin{aligned} \mu_{-2} &= \mu_{-D}(d_s)\mu_{-V}(v_s)\mu_{-A}(a_s) \\ \mu_{-1} &= \mu_{-D}(d_s)\mu_{-V}(v_s)\mu_{+A}(a_s) + \mu_{-D}(d_s)\mu_{+V}(v_s)\mu_{-A}(a_s) + \mu_{+D}(d_s)\mu_{-V}(v_s)\mu_{-A}(a_s) \\ \mu_{+1} &= \mu_{-D}(d_s)\mu_{+V}(v_s)\mu_{+A}(a_s) + \mu_{+D}(d_s)\mu_{-V}(v_s)\mu_{+A}(a_s) + \mu_{+D}(d_s)\mu_{+V}(v_s)\mu_{-A}(a_s) \\ \mu_{+2} &= \mu_{+D}(d_s)\mu_{+V}(v_s)\mu_{+A}(a_s) \\ A(u_{-1}) &= (2/3)\mu_{-1}l_u \quad ; \quad C(u_{-1}) = -l_u/3 \\ A(u_{+1}) &= (2/3)\mu_{+1}l_u \quad ; \quad C(u_{+1}) = l_u/3 \\ A(u_{-2}) &= \mu_{-2}(3L_u - 2l_u)/3 \\ C(u_{-2}) &= \frac{-[9(3L_u^2 - w^2) + l_u(6w - 13l_u)]}{18(3L_u - 2l_u)} \\ A(u_{+2}) &= \mu_{+2}(3L_u - 2l_u)/3 \\ C(u_{+2}) &= \frac{[9(3L_u^2 - w^2) + l_u(6w - 13l_u)]}{18(3L_u - 2l_u)} \end{aligned}$$

3 MATHEMATICAL MODEL

Upon substituting areas A 's and centers of areas C 's in Eq.(13), we have in inner cuboid i.e., (1,1,1)

$$\Delta u(k) = \frac{1}{54D} N_v v_s(k) + N_d d_s(k) + N_a a_s(k) \quad (14)$$

where

$$\begin{aligned} D &= l_d l_v l_a (3L_u + 4l_u) + [l_v d_s(k)a_s(k) + l_a d_s(k)v_s(k) + l_d a_s(k)v_s(k)](3L_u - 4l_u) \\ N_d &= [9(3L_u^2 - w^2) + l_u(6w - l_u)][3l_v l_a + v_s(k)a_s(k)] - 48l_u^2 v_s(k)a_s(k) \\ N_v &= [9(3L_u^2 - w^2) + l_u(6w - l_u)][3l_d l_a + a_s(k)d_s(k)] - 48l_u^2 d_s(k)a_s(k) \\ N_a &= [9(3L_u^2 - w^2) + l_u(6w - l_u)][3l_d l_v + v_s(k)d_s(k)] - 48l_u^2 v_s(k)d_s(k) \end{aligned}$$

$$\begin{aligned} \Delta u(k) &= l_u v_s(k)/(3l_v) \\ &\text{in cells (9,11,19),(12,14,17),(13,15,17),(16,10,19)} \\ &= \frac{(27L_u^2 - 9w^2 + 6l_u w)(v_s(k) + l_v) - l_u^2(l_v + 25v_s(k))}{18[3L_u(v_s(k) + l_v) - 4l_u v_s(k)]} \\ &\text{in cells (9,14,20),(16,15,20)} \\ &= \frac{(27L_u^2 - 9w^2 + 6l_u w)(d_s(k) - l_d) + l_u^2(l_d - 25d_s(k))}{18[3L_u((l_d - d_s(k)) + 4l_u d_s(k))]} \\ &\text{in cells (10,18,13),(11,18,12)} \\ &= l_u d_s(k)/(3l_d) \\ &\text{in cells (10,19,16),(11,19,9),(14,17,12),(15,17,13)} \\ &= \frac{(27L_u^2 - 9w^2 + 6l_u w)(v_s(k) - l_v) + l_u^2(l_v - 25v_s(k))}{18[3L_u((l_v - v_s(k)) + 4l_u v_s(k))]} \\ &\text{in cells (12,11,18),(13,10,18)} \\ &= \frac{(27L_u^2 - 9w^2 + 6l_u w)(d_s(k) + l_d) - l_u^2(l_d + 25d_s(k))}{18[3L_u((l_d + d_s(k)) - 4l_u d_s(k))]} \\ &\text{in cells (14,20,9),(15,20,16)} \\ &= \frac{(27L_u^2 - 9w^2 + 6l_u w)(a_s(k) + l_a) - l_u^2(l_a + 25a_s(k))}{18[3L_u(l_a + a_s(k)) - 4l_u a_s(k)]} \\ &\text{in cells (20,9,14), (20,16,15)} \\ &= \frac{-(27L_u^2 - 9w^2 + 6l_u w - 13l_u^2)}{18[3L_u - 2l_u]} \text{ in cells (18,18,18) } \quad (15) \\ &= \frac{(27L_u^2 - 9w^2 + 6l_u w - 13l_u^2)}{18[3L_u - 2l_u]} \text{ in cells (20,20,20) } \quad (16) \\ &= l_u/3 \text{ in cells (17,19,20), (19,20,17), (20,17,19)} \\ &= l_u a_s(k)/(3l_a) \text{ in cells (17,12,14), (17,13,15), (19,9,11), (19,16,10)} \\ &= \frac{(27L_u^2 - 9w^2 + 6l_u w)(a_s(k) - l_a) + l_u^2(l_a - 25a_s(k))}{18[3L_u(l_a - a_s(k)) + 4l_u a_s(k)]} \\ &\text{in cells (18,12,11), (18,13,10)} \\ &= -l_u/3 \text{ in cells (17,18,19), (18,19,17), (19,17,18)} \end{aligned}$$

4 PROPERTIES OF THE MODEL

In the previous section, the mathematical model, Eq. (14), has been presented for the fuzzy PID controller when all the three scaled inputs are inside the cuboid.

1. The control surface generated by $\Delta u(k)$ is continuous at any point in the 3D input space.
2. The incremental control output $\Delta u(k)$ increases as the distance of the point (d_s, v_s, a_s) from the origin in the 3D input space increases.
3. By comparing Eq. (14) with Eq. (1) one can recognize that fuzzy controller is very much similar in structure to the linear controller. Since N_d, N_v, N_a and D are nonlinear functions of scaled inputs $d_s(k), v_s(k)$ and $a_s(k)$, the fuzzy PID controller is a nonlinear PID controller with the variable gains given by $K_{Pd} = N_v/(54D); K_{id} = N_d/(54D); K_{Dd} = N_a/(54D)$.

In the case of linear PID controller the gains K_P^d, K_I^d and K_D^d

are constants.

- The minimum incremental control effort produced by the fuzzy controller is given by Eq. (15) while Eq. (16) gives the maximum incremental control effort.

Table 1. Outcomes of algebraic product AND and bounded sum OR

CELLS	μ_{-2}	μ_{-1}	μ_{+1}	μ_{+2}
Case 1: Inner cube defined by $-l_d \leq d_s \leq l_d, -l_v \leq v_s \leq l_v; -l_a \leq a_s \leq l_a.$				
(1,1,1)	(+)	(-)	(\times)	(\div)
Case 2: One input inside and two inputs outside the inner cuboid				
(20,16,15)(20,9,14)	0	0	-A	+A
(18,13,10)(18,12,11)	-A	+A	0	0
(17,13,15)(17,12,14) (19,16,10)(19,9,11)	0	-A	+A	0
(9,14,20)(16,15,20)	0	0	-V	+V
(12,11,18)(13,10,18)	-V	+V	0	0
(9,11,19)(12,14,17), (13,15,17)(16,10,19),	0	0	-V	+V
(15,20,16)(14,20,19)	0	0	-D	+D
(10,18,13)(11,18,12)	-D	+D	0	0
(11,19,9)(14,17,12) (15,17,13)(10,19,16)	0	0	-D	+D
Case 2: All three inputs outside the inner cuboid				
(20, 20, 20)	0	0	0	1
(17,19,20)(20,17,19)(19,20,17)	0	0	1	0
(17,18,19)(19,17,18)(18,19,17)	0	1	0	0
(18,18,18)	1	0	0	0

(+): (-D)(-V)(-A)

(-): (-D)(-V)(+A)+(-D)(+V)(-A)+(+D)(-V)(-A)

(\times): (-D)(+V)(+A)+(+D)(-V)(+A)+(+D)(+V)(-A)

(\div): (+D)(+V)(+A)

5 CONCLUSIONS

In this paper, a mathematical model for a fuzzy PID controller has been derived using L-type, Γ -type and Π -type fuzzy

membership functions, algebraic product triangular norm, bounded sum triangular co-norm, algebraic product inference method, modified output membership functions and CoSdefuzzification method. The model obtained is shown to be nonlinear whereas the model (Class I model) in Mohan and Sinha (2006) was indeed a linear model. The (modified) membership functions considered in Fig. 3 could lead to this difference in the end result.

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